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Corrections to "Mathematical Models for Cochannel Interference in FH/MFSK Multiple Access Systems"

T.-Y. YAN AND C. C. WANG

In the above paper,[1] the authors misrepresented the approach taken by Geraniotis and Pursley ([13] in the paper). Their approach does *not* require an independence assumption, and so the statement, "Geraniotis and Pursley also use the independence assumption to evaluate the error probabilities for slow FH/MFSK over fading channels," should be deleted from our introduction.

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Collision Resolution Protocols Utilizing Absorptions and Collision Multiplicities

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***Abstract*—In this correspondence, we consider the random accessing of a single slotted channel by a large number of packet transmitting users, whose cumulative traffic is Poisson. We assume the existence of the same feedback as that of the MCRAI protocols of Georgiadis, and full channel sensing, and we develop collision resolution algorithms that utilize the absorption concept of Gallager. We observe the improvement in the throughputs induced by the absorption, as well as the improved delay characteristics. Finally, we draw some conclusions about the limitations of the absorption idea.**

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I. INTRODUCTION

We consider the accessing of a single, errorless, slotted channel by a Poisson packet traffic. For this channel and user model, several transmission algorithms have been proposed [1]–[3], where the existence of feedback information is always assumed. In this paper, we assume the increased feedback information as in [6], and utilizing the absorption concept of Gallager [4], we design and analyze algorithms whose performance is better than the performance induced by the algorithms in [6].

II. THE ALGORITHMS WITH ABSORPTION—GENERAL OPERATION

We assume the same channel and user model as in [6]. The users sense the feedback continuously, and the feedback distinguishes packet collision multiplicities exactly up to order K. For packet collision multiplicities of order greater than K, the feedback informs the users that at least $K + 1$ packets have collided.

The description of the algorithm is facilitated if we decouple the arrival axis from the channel axis. The arrival axis contains points which correspond to packet arrival instants, and it is segmented into consecutive, possibly overlapping intervals of length $\Delta(K)$ or less, where $\Delta(K)$ is a parameter to be optimized later. The channel axis is segmented into consecutive nonoverlapping intervals, whose lengths are integral multiples of a slot duration. From now on, when we refer to some time instant t, it will correspond to time on the arrival axis. The channel axis represents real time, and every reference to some time instant T will correspond to time on this axis.

The algorithm operates in *sessions*. Let us assume that when a session begins, at some time T, all packets generated prior to t have been successfully transmitted. The interval $T - t$ is called the *unexamined* arrival interval at T, while the interval $[0, t)$ is called the *resolved* arrival interval at T. At the beginning of this session, an interval $[t, t + \Delta)$ is initially examined ($\Delta = \min(T - t, \Delta(K))$ (Fig. 1). During the resolution process of $(t, t + \Delta)$, the interval is possibly split into a number of smaller intervals, each one of which joins a queue, and awaits its turn to be examined by the algorithm. We denote these intervals by $[I_b(R), I_e(R))$; $R \geqslant 0$. They lie on the arrival axis with beginning and end points $I_b(R)$ and $I_e(R)$, respectively. The index R indicates their position in the queue, and it specifies the order of their service by the channel. The interval which is currently served is the one which occupies position 0 of the queue [interval $[I_b(0), I_e(0))$].

At the end of each slot, the users are informed about the number of packets transmitted over the slot through the available feedback. Let us denote by F_T, $T = 1, 2, \cdots$, the value of the feedback information, which the users observe at time T (end of slot $[T - 1, T]$). We assume that F_T takes the value i, $0 \leqslant i \leqslant K$, if i packets were transmitted over slot $[T - 1, T]$, while F_T takes the value e, if the number of packets transmitted over the slot $[T - 1, T]$ exceeded the upper detectable limit K of collision multiciplicities.

Let us denote by $N(R)$, $R \geqslant 0$, the actual number of packets contained in the interval $[I_b(R), I_e(R))$. We also denote by $\hat{N}(R)$, $R \geqslant 0$, an estimate from below of $N(R)$ ($\hat{N}(R) \leqslant N(R)$), which the users obtain based on the available feedback information. The description of the algorithm is facilitated by the introduction of a parameter $P(R)$, $R \geqslant 0$, which is defined

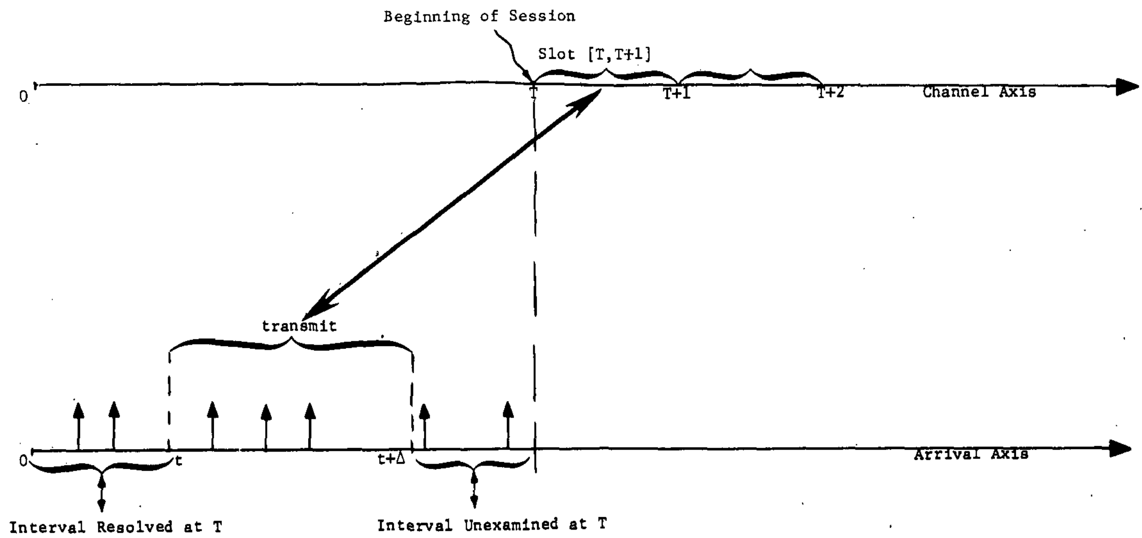

Fig. 1.

as follows: $P(R)$ is equal to 0 iff $\hat{N}(R) = N(R)$, while $P(R)$ is equal to 1 iff $\hat{N}(R) < N(R)$.

At each time instant during the operation of the algorithm, and for every interval $[I_b(R), I_e(R))$ in the queue, the users know i) its two endpoints $I_b(R)$ and $I_e(R)$, ii) the value (if any) of $\hat{N}(R)$, and iii) the value (if any) of $P(R)$. The actual number of packets $N(R)$, contained in $[I_b(R), I_e(R))$, is known to the users, iff $P(R) = 0$.

We now distinguish the following cases.

1) $P(R) = 0$ or equivalently $\hat{N}(R) = N(R)$. Then, we say that $[I_b(R), I_e(R))$ is in state $S(K, N(R))$ or equivalently in state $S(K, \hat{N}(R))$. K, as before, is the upper detectable limit of collision multiplicities.

2) $P(R) = 1$ or equivalently $\hat{N}(R) < N(R)$. Then, we say that $[I_b(R), I_e(R))$ is in state $S(K, \hat{N}(R), N(R))$. K is as in case 1.

During its operation, the algorithm utilizes the following parameters.

a) The above defined parameters $K, I_b(R), I_e(R), F_T, \hat{N}(R), P(R)$.

b) A global counter CR, which is updated according to the rules of the algorithm. CR is set to 1 at the beginning of a new session and when it first takes the value 0, it signals the end of the session.

c) For every value of K a set of parameters σ_N $(2 \leq N \leq K)$ and a parameter $\Delta(K)$, whose optimal values are shown in Tables I and II, respectively (for further details about the optimization procedure, see [6] and [8]).

d) A parameter M, which at the beginning of steps 4 and 5 is set equal to the value of $\hat{N}(0)$.

e) A parameter $I(0)$, which at the beginning of steps 4 and 5 is set equal to the length of the interval $[I_b(0), I_e(0))$.

f) A parameter T, whose value corresponds to a real-time clock reading.

g) A parameter t, whose value at the beginning of a new session (beginning of step 1) corresponds to the right endpoint of the resolved arrival interval $[0, t]$. The algorithm begins from step 0.

0) $T = 1$
$t = 0$
go to step 1

1) $\Delta = \min(T - t, \Delta(K))$
$I_b(0) = t$
$I_e(0) = t + \Delta$
$t \to t + \Delta$
CR = 1
go to step 2

TABLE I
SPLITTING PARAMETER σ_N

N	σ_N
2	0.5
3	0.411972
4	0.342936
5	0.288214
6	0.249289
7	0.219964
8	0.196794

TABLE II
THROUGHPUT BOUNDS

K	$x=\lambda\Delta(K)$	$\lambda_l(K)$	$\lambda_u(K)$	MCRAI
1	1.266	.48711*	.48711*	.462**
2	1.453	.51426	.51426	.4926***
3	1.600	.52461	.52461	.5159***
4	1.720	.52901	.52901	.5257***
5	1.807	.53077	.53077	.5297***
6	1.855	.53137	.53137	.5311***
7	1.876	.53154	.53154	.5315***
8	1.882	.53159	.53159	-

* Gallager's algorithm (ternary feedback with absorbtion)
** Massey's algorithm (ternary feedback with skip step)
*** MCRAI throughputs with K energy detectors. See [6]

2) All users with a packet in $[I_b(0), I_e(0))$ transmit in slot $[T, T+1]$

$T \to T+1$
If $F_T = 0$ or 1 set $\hat{N}(0) = F_T$, $P(0) = 0$; go to step 3
If $2 \leqslant F_T \leqslant K$ set $\hat{N}(0) = F_T$, $P(0) = 0$; go to step 4
If $F_T = e$ set $\hat{N}(0) = K$, $P(0) = 1$; go to step 5

3) The interval $[I_b(0), I_e(0))$ is either in state $S(K, 0)$ or in state $S(K, 1)$

$\mathrm{CR} \to \mathrm{CR} - 1$
If CR = 0 the session has ended; go to step 1
If $\mathrm{CR} \neq 0$

$$\left.\begin{aligned} I_b(R-1) &= I_b(R) \\ I_e(R-1) &= I_e(R) \\ \hat{N}(R-1) &= \hat{N}(R) \\ P(R-1) &= P(R) \end{aligned}\right\}; \quad R = 1, \cdots, \mathrm{CR}$$

If $\hat{N}(0) = 0$ and $P(0) = 0$ go to step 3
If $\hat{N}(0) = 1$ and $P(0) = 0$ go to step 2
If $2 \leqslant \hat{N}(0) \leqslant K$ and $P(0) = 0$ go to step 4
If $\hat{N}(0) = 0$ and $P(0) = 1$ go to step 2
If $1 \leqslant \hat{N}(0) \leqslant K$ and $P(0) = 1$ go to step 5

4) The interval $[I_b(0), I_e(0))$ is in state $S(K, \hat{N}(0))$

Let $M = \hat{N}(0)$
Let $I(0) = I_e(0) - I_b(0)$
$I_e(0) = I_b(0) + \sigma_M \cdot I(0)$; σ_M splitting parameter, $0 < \sigma_M < 1$

$$\left.\begin{aligned} I_b(R+1) &= I_b(R) \\ I_e(R+1) &= I_e(R) \\ \hat{N}(R+1) &= \hat{N}(R) \\ P(R+1) &= P(R) \end{aligned}\right\}; \quad R = \mathrm{CR} - 1, \cdots, 1$$

$I_b(1) = I_b(0) + \sigma_M \cdot I(0)$
$I_e(1) = I_b(0) + I(0)$
$\mathrm{CR} \to \mathrm{CR} + 1$

All users with a packet in $[I_b(0), I_e(0))$ transmit in slot $[T, T+1]$

$T \to T+1$
$\hat{N}(1) = M - F_T$
$P(1) = 0$
If $F_T = 0$ or 1 set $\hat{N}(0) = F_T$, $P(0) = 0$; go to step 3
If $2 \leqslant F_T \leqslant K$ set $\hat{N}(0) = F_T$, $P(0) = 0$; go to step 4

5) The interval $[I_b(0), I_e(0))$ is in state $S(K, \hat{N}(0), N(0))$

Let $M = \hat{N}(0)$
Let $I(0) = I_e(0) - I_b(0)$
$I_e(0) = I_b(0) + 2^{-1}I(0)$

$$\left.\begin{aligned} I_b(R+1) &= I_b(R) \\ I_e(R+1) &= I_e(R) \\ \hat{N}(R+1) &= \hat{N}(R) \\ P(R+1) &= P(R) \end{aligned}\right\}, \quad R = \mathrm{CR} - 1, \cdots, 1$$

$I_b(1) = I_b(0) + 2^{-1}I(0)$
$I_e(1) = I_b(0) + I(0)$
$\mathrm{CR} \to \mathrm{CR} + 1$

All users with a packet in $[I_b(0), I_e(0))$ transmit in slot $[T, T+1]$

$T \to T+1$
If $F_T \leqslant M$ set $\hat{N}(0) = F_T$, $P(0) = 0$; go to step 5a
If $M < F_T \leqslant K$ set $\hat{N}(0) = F_T$, $P(0) = 0$; go to step 5b
If $F_T = e$ set $\hat{N}(0) = K$, $P(0) = 1$; go to step 5b

5a) $\hat{N}(1) = M - \hat{N}(0)$
$P(1) = 1$
If $\hat{N}(0) = 0$ or 1 and $P(0) = 0$ go to step 3
If $2 \leqslant \hat{N}(0) \leqslant K$ and $P(0) = 0$ to to step 4

5b) $[I_b(1), I_e(1)]$ is absorbed in the unexamined interval
$t \to t - 2^{-1}I(0)$

$$\left.\begin{aligned} I_b(R-1) &= I_b(R) \\ I_e(R-1) &= I_e(R) \\ \hat{N}(R-1) &= \hat{N}(R) \\ P(R-1) &= P(R) \end{aligned}\right\}; \quad R = 2, \cdots, \mathrm{CR} - 1$$

$\mathrm{CR} \to \mathrm{CR} - 1$

TABLE III
EXPECTED DELAYS

λ	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8
.1	0.283	0.273	0.264	0.263	0.263	0.263	0.263	0.263
.2	0.827	0.747	0.739	0.734	0.732	0.728	0.728	0.728
.35	3.540	2.804	2.666	2.567	2.584	2.524	2.524	2.498
.45	14.304	8.379	8.070	7.131	7.108	7.070	7.035	6.893
.5	-	26.000	20.749	19.665	19.523	18.552	18.517	18.133
.52	-	-	31.695	31.439	31.326	30.761	30.523	30.202

If $M < \hat{N}(0) \leqslant K$ and $P(0) = 0$ go to step 4
If $\hat{N}(0) = K$ and $P(0) = 1$ go to step 5

III. Analysis—Performance Evaluation

We analyzed the algorithms presented in the previous section. We found that they are stable in the region $[0, \lambda(K)]$. We found upper $(\lambda_u(K))$ and lower $(\lambda_l(K))$ bounds for $\lambda(K)$. In Table II we list $\lambda_l(K)$, $\lambda_u(K)$ for K ranging from 1 to 8. The optimal, initially examined arrival interval of a session, $\Delta(K)$, is also given in the same table. Actually $x \triangleq \lambda\Delta(K)$ is provided. We also include in Table II the throughput of the MCRAI algorithm discussed in [6]. For further details, associated with the algorithm analysis and the evaluation of its performance, the interested reader is referred to [8].

IV. Conclusions

We simulated the algorithms with absorptions for different K and λ values. We used the simulations to compute expected per-packet delays, where delay is defined as the time (in slot units) between the arrival and the successful transmission of a packet. We exhibit our results in Table III. Comparing this table to [6, Table IV], we observe that the gain in delays via the algorithms with absorptions becomes minimal, after $K = 5$. Now comparing the throughput results (Table II), we see that the highest gain in throughput, by using algorithms with absorptions, is attained when $K = 1$ (Gallager algorithm versus Massey algorithm) and $K = 2$. From then on, the gain decreases monotonically and it becomes negligible when $K > 7$. Therefore, as the available feedback information increases, the advantages of the absorption concept become negligible. By observing Table II, we point out that algorithms with absorptions can achieve almost the same performance (throughput) with the algorithms without absorptions with one less energy detector. We point out that as the available feedback information increases, so does the sensitivity to channel errors of the algorithms with absorptions. If channel errors occur, then a different class of algorithms may have to be devised.

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The Optimal Retry Distribution for Lightly Loaded Slotted Aloha Systems

B. SIMON AND L. G. VOTTA

Abstract—**Most of the analytical work on slotted Aloha has been concerned with maximizing the utilization. The delay experienced by a transmission has not been a primary issue. In this note we take the opposite point of view. It is assumed that the system is operating far below capacity (as is typical in practice), and we concern ourselves with the problem of minimizing packet delay in the event of a collision. The retry distribution that minimizes the average delay is derived. Surprisingly, the optimal retry distribution has finite support, and in fact, one never waits more than three slots before retransmitting.**

I. Introduction

Slotted Aloha is among the simplest and oldest multiple access protocols for communication systems, and is a viable access technique in certain applications (e.g., a system supporting a large number of light users). There are a few basic analytical results for slotted Aloha systems, the most famous being the theoretical maximum traffic intensity of e^{-1} for the standard infinite source model. The common derivations of this result implicitly assume that colliding packets retransmit in the distant future. As many authors have pointed out [2], [3], one can run into problems when colliding packets do not wait forever to retransmit. The trouble is that the slotted Aloha protocol is no longer stable, even for small traffic intensities, unless it is very carefully controlled.

More recent studies have shown that slotted Aloha can be controlled in such a way that it remains stable [2], [5], and that it is possible to obtain a utilization in excess of e^{-1} [1], [6], [7]. In most of these models, though, the delay experienced by a packet is not a primary issue. In this note, we take the opposite point of view. We assume that the system is operating far below capacity (as is typical in practice), and concern ourselves with the problem of minimizing packet delay in the event of a collision.

In the next section we will find the optimal retry distribution for an idealized slotted Aloha system. Although the system we optimize for is idealized, it closely resembles a slotted Aloha system that is lightly loaded. In a lightly loaded system, collisions involving more than two packets are very rare. Our model assumes that there is never a collision involving more than two packets.

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II. Main Result

Consider a slotted Aloha system designed in such a way that collisions involving more than two packets never occur. When two packets do collide, they each independently choose an integer m from a retry distribution, $\{p_n\}$, and wait m slots before retransmitting. If they choose the same integer, they will recollide, and start the process again. It is further assumed that new packets will not arrive while a collision is being resolved. Any slotted Aloha system in light traffic will behave approximately in this manner. Our goal is to find the retry distribution that minimizes the average delay experienced by colliding packets.

Let p_j, $j = 1, 2, \cdots$, be a retry distribution. Define $A(p)$ to be the expected delay experienced by an unsuccessful transmission (i.e., the average number of slots from the time the transmission is first attempted, until it is successfully transmitted). Then

$$A(p) = \frac{\sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} (j+k) p_j p_k + \sum_{j=1}^{\infty} j p_j^2}{1 - \sum_{j=1}^{\infty} p_j^2}. \tag{1}$$

Theorem: The optimal retry distribution is

$$P^* = \left\{ \frac{4-\sqrt{6}}{3}, \frac{1}{3}, \frac{\sqrt{6}-2}{3}, 0, 0, \cdots \right\}.$$

Proof: We prove that P^* is optimal by showing that it is a Kuehn–Tucker point [4] in the set of probability distribution on the integers. The Kuehn–Tucker equations are

$$\frac{\partial A(p)}{\partial p_i} - \lambda - \mu_i = 0, \qquad i = 1, 2, \cdots \tag{2}$$

$$\mu_i p_i = 0, \qquad i = 1, 2, \cdots \tag{3}$$

$$\mu_i \geqslant 0, \qquad i = 1, 2, \cdots \tag{4}$$

$$\sum_{i=1}^{\infty} p_i = 1. \tag{5}$$

Taking derivatives, we have

$$\frac{\partial A(p)}{\partial p_n} = \frac{n + \sum_{j=1}^{\infty} j p_j + 2A(p) p_n}{1 - \sum_{j=1}^{\infty} p_j^2} \tag{6}$$

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